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QUANTUM MECHANICS IN GENERALIZED HILBERT SPACE

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CONTENTS

Synopsis.

An attempt is made to generalize the Hilbert space of quantum mechanics in analogy with the development of the general relativity theory from the theory of special relativity. The state vectors, ψ, $\overline{\psi}$, of quantum mechanics are found to be analogous to the four-velocity, v^{μ}, of relativity and therefore coordinates, χ, $\overline{\chi}$, are introduced, corresponding to the coordinates x^{μ} of a particle, such that the time derivatives of χ and $\overline{\chi}$ equal ψ and $\overline{\psi}$. The metric η, used in constructing the probability density, is supposed to be a function of χ and $\overline{\chi}$. The unitary transformations of the usual theory are replaced by quite general transformations χ and $\overline{\chi}$. A tensor calculus for this generalized Hilbert space is developed and equations of motion for the states and the dynamical variables are postulated as generalizations of the usual Heisenberg equations when the ordinary time differentiation is replaced by invariant time differentiation. In this way a non-linear theory is obtained. However, the expectation values of the dynamical variables are found to be the same in the new theory as in the old, showing that this theory cannot give any physical results different from those of the usual theory.

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1. Introduction.

The present-day quantum mechanics has been successful in explaining a large number of phenomena, particularly those involving electrons and electromagnetic radiation. It has, however, not been so successful in dealing with other particles. The discovery of several new particles in recent years seems to indicate that the basis of the present theory ought to be broadened. In an ideal theory, one should be able to describe the various particles as possible states of one system. It is probable that this can be achieved by constructing a non-linear theory in which the principle of superposition of states is valid only as a first approximation.

Some attempts in this direction have recently been made, notably by SCHIFF (1951 a, b, 1952), by THIRRING (1952), by HEISENBERG (1953, 1954) and by HEISENBERG, KORTEL and MITTER (1955), who introduced non-linear terms into the wave equations. The addition of such terms is, however, an entirely arbitrary procedure and therefore unsatisfactory. These attempts can therefore be considered only as phenomenological until they have some acceptable principles as their basis.

A well-known example of a non-linear theory in classical physics is the theory of general relativity. The special relativity theory allows only linear transformations of the coordinates; the general theory abandons this restriction and takes quite general coordinate transformations into account. This leads in a fairly natural way to the explanation of the gravitational phenomena. But gravitation plays only a very minor role in atomic and nuclear phenomena and therefore the theory of general relativity in itself is not of much interest to the atomic physicist. However, one can still learn a great deal from it. Its methods may, for instance, be applied to the construction of a more general Hilbert space in which the unitary transformations of the usual theory can be abandoned in favour of more general transformations. This paper deals with exploring this possibility. It is shown here that such a generalization is possible and leads, as expected, to non-linear wave equations in quantum mechanics.

The development outlined below is similar to that of the general relativity theory. However, it is hoped that this paper can be understood, at least in its main line of arguments, without previous familiarity with general relativity or Riemannian geometry.

2. Analogy between Relativity Theory and Quantum Mechanics.

We here start by discussing a Hilbert space of finite dimensions, N. A system in quantum mechanics is completely specified when the components, ψ^m, of its state vector are known in all the N mutually orthogonal directions in Hilbert space. The state ψ is usually normalized to unity, which means that

$$\sum_{m=1}^{N} \overline{\psi}^m \psi^m = 1 . \tag{2.1}$$

Here $\overline{\psi}^m$ is the complex conjugate of ψ^m. One could, if one wished, choose a different normalization for ψ, but normalization to unity is most convenient. The unitary transformations are such that they leave (2.1) invariant. Indicating the transformed variables by primes, we have

$$\sum_{m} \overline{\psi}^{m'} \psi^{m'} = \sum_{m} \overline{\psi}^m \psi^m = 1 . \tag{2.2}$$

If we define

$$\psi_m = \overline{\psi}^m \tag{2.3}$$

we can write (2.1) as

$$\sum_{m} \psi_m \psi^m = 1 . \tag{2.4}$$

Let us denote a general dynamical variable by A with components $A_{\overline{m}n}$. When ψ goes over to ψ' by means of a unitary transformation, A goes over to A' such that the expression $\sum_{m,\,n} \bar{\psi}^m A_{\bar{m}n} \psi^n$ remains invariant:

$$\sum_{m,\,n} \bar{\psi}^{m'} A'_{\bar{m}n} \psi^{n'} = \sum_{m,\,n} \bar{\psi}^m A_{\bar{m}n} \psi^n. \tag{2.5}$$

The reason why we have put a bar over m in $A_{\bar{m}n}$ is that this suffix is contracted with $\bar{\psi}^m$ while the other suffix, n, which is without a bar, is contracted with ψ^n.

With the help of the above notation for the components of a dynamical variable, we can write (2.3) as

$$\psi_m = \sum_n \bar{\psi}^n \eta_{\bar{n}m}, \tag{2.6}$$

where $\eta_{\bar{n}m}$ is the unit matrix.

In special relativity we meet an analogous situation. If v^μ denotes the four-velocity, $\frac{dx^\mu}{d\tau}$, of a particle (τ is the proper time, $c = 1$), we have

$$v_\mu v^\mu = 1. \tag{2.7}$$

Here the covariant components, v_μ, are related to the contravariant vector v^μ by means of the metric $g_{\nu\mu}$:

$$v_\mu = \sum_\nu v^\nu g_{\nu\mu}, \tag{2.8}$$

where

$$\left.\begin{aligned} g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \\ g_{\mu\nu} = 0 \ \text{ for } \ \mu \neq \nu. \end{aligned}\right\} \tag{2.9}$$

If one expressed x^μ as functions of some other parameter s, one would get another factor instead of 1 on the right-hand side of (2.7). However, it is most convenient to have the normalization 1 by choosing the independent variable as τ.

We now notice a formal similarity between the equations (2.4) and (2.6) of the quantum theory, on the one hand, and the equations (2.7) and (2.8) of relativity, on the other. The analogue of the equation (2.5) in relativity would merely specify the transformation properties of a second rank tensor.

Because of this formal similarity between the relativity and the quantum theories, we can say that the quantum state ψ with components ψ^m corresponds to the relativistic velocity $\vec{v}$ with components v^μ.

3. Coordinates in Hilbert Space and Generalization of η.

The fact that ψ^m corresponds to the velocity v^μ suggests that we introduce coordinates χ^m such that, by definition,

$$\psi^m = \frac{d\chi^m}{dt}. \tag{3.1}$$

This relation is analogous to the definition

$$v^\mu = \frac{dx^\mu}{d\tau}.$$

The χ^m's do not form a vector, just as x^μ does not constitute a vector in relativity. The upper position of the index m in χ^m is inserted only for convenience and does not imply that it is a vector.

From (2.4) and (3.1) it follows that

$$d\chi_m\, d\chi^m = dt^2, \tag{3.2}$$

where, according to (2.6),

$$d\chi_m = \sum_n d\overline{\chi}^n\, \eta_{\overline{n}m}. \tag{3.3}$$

(3.2) can also be written as

$$\sum_{m,\,n} \eta_{\overline{m}n}\, d\overline{\chi}^m\, d\chi^n = dt^2 \tag{3.4}$$

which is analogous to the relativity relation

$$\sum_{\mu,\,\nu} g_{\mu\nu}\, dx^\mu\, dx^\nu = d\tau^2. \tag{3.5}$$

In special relativity the $g_{\mu\nu}$'s are constants given by (2.9). The transition from this theory to the theory of general relativity consists in abandoning the constancy of $g_{\mu\nu}$ and allowing them to be functions of the coordinates x^{μ}. In view of the formal similarity between the equations (3.4) and (3.5), it now suggests itself that in quantum theory we regard $\eta_{\overline{m}n}$ as functions of χ and $\overline{\chi}$:

$$\eta_{\overline{m}n} = \eta_{\overline{m}n}(\chi, \overline{\chi}). \tag{3.6}$$

4. Definitions of Vectors and Tensors.

We now assume that quite general transformations of χ and $\overline{\chi}$ are possible such that the transformed coordinates $\chi^{m'}$ depend on $\chi^1, \chi^2, \chi^3, \ldots\ldots\ldots \chi^N$ and similarly $\overline{\chi}^{m'}$ depend on $\overline{\chi}^1, \overline{\chi}^2, \overline{\chi}^3, \ldots\ldots\ldots \overline{\chi}^N$:

$$\left.\begin{aligned} \chi^{m'} &= \chi^{m'}(\chi^1, \chi^2, \chi^3, \ldots \chi^N) \\ \overline{\chi}^{m'} &= \overline{\chi}^{m'}(\overline{\chi}^1, \overline{\chi}^2, \overline{\chi}^3, \ldots \overline{\chi}^N) \\ & \qquad\qquad m = 1 \ldots N. \end{aligned}\right\} \tag{4.1}$$

Note that $\chi^{m'}$ does not depend on $\overline{\chi}^r$, nor does $\overline{\chi}^{m'}$ depend on χ^r.

From (4.1) it follows that

$$d\chi^{m'} = \frac{\partial \chi^{m'}}{\partial \chi^n} d\chi^n \tag{4.2 a}$$

and

$$d\overline{\chi}^{m'} = \frac{\partial \overline{\chi}^{m'}}{\partial \overline{\chi}^n} d\overline{\chi}^n. \tag{4.2 b}$$

We now follow the usual convention that, unless otherwise stated, when a suffix occurs once below and once above, summation over it will be understood.

We define a 'contravariant vector' as one whose components transform like $d\chi^m$ and a 'conjugate contravariant vector' as one whose components transform as $d\overline{\chi}^m$. Sometimes, we distinguish vectors of the kind $d\chi^m$ by calling them 'ordinary' as contrasted with conjugate vectors. Thus, an ordinary contravariant vector, A^m, transforms as

$$A^{m'} = \frac{\partial \chi^{m'}}{\partial \chi^n} A^n \tag{4.3 a}$$

and a conjugate contravariant vector, $A^{\overline{m}}$, as

$$A^{\overline{m}'} = \frac{\partial \overline{\chi}^{m'}}{\partial \overline{\chi}^{n}} A^{\overline{n}}. \tag{4.3 b}$$

We put a bar over the suffix which transforms as a conjugate vector, while the suffixes which transform as ordinary vectors will be left unbarred.

From (4.2 a, b) and (3.1) it follows that ψ is an ordinary and $\overline{\psi}$ a conjugate contravariant vector.

Besides contravariant vectors, we also have covariant vectors. An ordinary covariant vector, A_m, is defined to transform as

$$A'_m = \frac{\partial \chi^{n}}{\partial \chi^{m'}} A_n. \tag{4.4 a}$$

The conjugate covariant vectors transform as

$$A'_{\overline{m}} = \frac{\partial \overline{\chi}^{n}}{\partial \overline{\chi}^{m'}} A^{\overline{n}}. \tag{4.4 b}$$

These definitions are arranged so that, by contracting the indices of a covariant vector and a contravariant vector of the same kind, we get an invariant result:

$$A'_m B^{m'} = A_m B^m \tag{4.5 a}$$

$$A'_{\overline{m}} B^{\overline{m}'} = A_{\overline{m}} B^{\overline{m}}. \tag{4.5 b}$$

Tensors of higher ranks can be defined in exactly the same way as in the ordinary tensor analysis. Thus, a second rank tensor $A^{m}{}_{n}$ transforms as

$$A^{m}{}_{n}{}' = \frac{\partial \chi^{m'}}{\partial \chi^{a}} \frac{\partial \chi^{b}}{\partial \chi^{n'}} A^{\overline{a}}{}_{b} \tag{4.6 a}$$

and a tensor $A^{\overline{m}}{}_{n}$ transforms as

$$A^{\overline{m}}{}_{n}{}' = \frac{\partial \overline{\chi}^{m'}}{\partial \overline{\chi}^{a}} \frac{\partial \chi^{b}}{\partial \chi^{n'}} A^{\overline{a}}{}_{b}, \tag{4.6 b}$$

etc.

From (3.4) and (4.2 a, b), it is clear that, if we have

$$\eta'_{\overline{m}n} = \frac{\partial \overline{\chi}^a}{\partial \overline{\chi}^{m'}} \frac{\partial \chi^b}{\partial \chi^{n'}} \eta_{\overline{a}b}, \tag{4.7}$$

the expression (3.4) will be invariant. We can say that η is a covariant tensor of the mixed kind. It is easy to see that $A_{\overline{m}n}$ in (2.5) is also a covariant tensor, of the mixed kind.

Besides being a tensor, the equation (3.4) shows that η is a Hermitian matrix, i. e.,

$$\overline{\eta}_{\overline{m}n} = \eta_{\overline{n}m}. \tag{4.8}$$

5. Covariant Differentiation.

If φ is a scalar, i. e., if

$$\varphi' = \varphi,$$

it follows that

$$\frac{\partial \varphi'}{\partial \chi^{m'}} = \frac{\partial \varphi}{\partial \chi^a} \frac{\partial \chi^a}{\partial \chi^{m'}}. \tag{5.1}$$

Comparing (5.1) with (4.4 a) we see that the gradient, $\frac{\partial \varphi}{\partial \chi^m}$, is a covariant vector. Similarly one can see that $\frac{\partial \varphi}{\partial \overline{\chi}^m}$ is a conjugate covariant vector.

Let us now consider the gradient of a vector A^m. We have, on using (4.3 a),

$$\left.\begin{aligned} \frac{\partial A^{m'}}{\partial \chi^{n'}} &= \frac{\partial}{\partial \chi^{n'}} \left[\frac{\partial \chi^{m'}}{\partial \chi^a} A^a \right] \\ &= \frac{\partial \chi^b}{\partial \chi^{n'}} \frac{\partial}{\partial \chi^b} \left[\frac{\partial \chi^{m'}}{\partial \chi^a} A^a \right] \\ &= \frac{\partial \chi^{m'}}{\partial \chi^a} \frac{\partial \chi^b}{\partial \chi^{n'}} \frac{\partial A^a}{\partial \chi^b} + \frac{\partial \chi^b}{\partial \chi^{n'}} \frac{\partial^2 \chi^{m'}}{\partial \chi^b \partial \chi^a} A^a. \end{aligned}\right\} \tag{5.2}$$

Comparing (5.2) with (4.6 a) we see that $\frac{\partial A^m}{\partial \chi^n}$ would have been a tensor if the last term in (5.2) were absent. Because of its presence, $\frac{\partial A^m}{\partial \chi^n}$ is no longer a tensor. As in general relativity, we therefore introduce an 'affinity' Γ^m_{rn} such that, by definition,

$$A^m_{;\,n} = \frac{\partial A^m}{\partial \chi^n} + A^r\,\Gamma^m_{rn} \tag{5.3}$$

is a tensor. We call $A^m_{;\,n}$ the covariant derivative of A^m and denote it by a semi-colon. It is evident that Γ^m_{rn} cannot be a tensor. We shall find its transformation properties presently.

Since $A^m_{;\,n}$ is by definition a tensor, we have, on using (4.6 a),

$$A^{m\,\prime}_{;\,n} = \frac{\partial \chi^{m'}}{\partial \chi^a}\,\frac{\partial \chi^b}{\partial \chi^{n'}}\,A^a_{\;;\,b}.$$

When we substitute the definition (5.3) of $A^m_{;\,n}$ and express $A^{m'}$ in terms of A^r by using (4.3 a), we get

$$\Gamma^{m\,\prime}_{nr} = \frac{\partial \chi^{m'}}{\partial \chi^a}\,\frac{\partial \chi^b}{\partial \chi^{n'}}\,\frac{\partial \chi^c}{\partial \chi^{r'}}\,\Gamma^a_{bc} + \frac{\partial \chi^{m'}}{\partial \chi^a}\,\frac{\partial^2 \chi^a}{\partial \chi^{n'}\,\partial \chi^{r'}}. \tag{5.4}$$

This is precisely the transformation law for Γ^m_{nr} in general relativity. Note that, because of the second term in the right-hand side of (5.4), Γ^m_{nr} is not a tensor.

It is easy to see that, since the last term in (5.4) is symmetric in n and r, Γ^m_{nr} will remain symmetric in all coordinate frames if it is chosen symmetric in one. This, of course, does not *prove* that Γ^m_{nr} is symmetric. In this paper we shall take it to be symmetric for the sake of simplicity.

We have seen that the gradient of a scalar is a vector. We can therefore say that the covariant derivative of a scalar is the same as the ordinary derivative

$$\varphi_{;\,r} = \frac{\partial \varphi}{\partial \chi^r}. \tag{5.5}$$

Assume now that the usual product rule for differentiation holds also for covariant differentiation

$$(fg)_{;\,n} = f_{;\,n}\, g + f g_{;\,n} \tag{5.6}$$

so that, in particular,

$$(A^m B_m)_{;\,n} = A^m_{;\,n} B_m + A^m B_{m;\,n}. \tag{5.7}$$

Since $A^m B_m$ is a scalar, we get from (5.5)

$$(A^m B_m)_{;\,n} = \frac{\partial A^m}{\partial \chi^n} B_m + A^m \frac{\partial B^m}{\partial \chi^n}.$$

When we substitute this and (5.3) in (5.7), we find that

$$A_{m;\,n} = \frac{\partial B_m}{\partial \chi^n} - B_a \Gamma^a_{mn}. \tag{5.8}$$

This provides the rule for differentiation of covariant vectors.

The rules for differentiating conjugate vectors with regard to $\bar{\chi}^n$ are similar. One has there to use an affinity which is the complex conjugate of Γ^m_{nr}.

$$B^{\overline{m}}_{;\,\overline{n}} = \frac{\partial A^{\overline{m}}}{\partial \bar{\chi}^n} + A^{\overline{a}} \Gamma^{\overline{m}}_{\overline{a}\,\overline{n}} \tag{5.9}$$

$$B_{\overline{m};\,\overline{n}} = \frac{\partial B_{\overline{m}}}{\partial \bar{\chi}^n} - B_{\overline{a}} \Gamma^{\overline{a}}_{\overline{m}\,\overline{n}}, \tag{5.10}$$

where

$$\Gamma^{\overline{m}}_{\overline{r}\,\overline{n}} = \overline{\Gamma^m_{rn}}. \tag{5.11}$$

So far the discussion has been quite analogous to that of the usual tensor analysis. Let us now consider the differential coefficient of a conjugate vector with respect to χ^n. We have

$$\left.\begin{aligned} \frac{\partial A^{\overline{m}'}}{\partial \chi^{n'}} &= \frac{\partial}{\partial \chi^{n'}} \left[\frac{\partial \bar{\chi}^{m'}}{\partial \bar{\chi}^a} A^{\overline{a}} \right] \\ &= \frac{\partial \chi^b}{\partial \chi^{n'}} \frac{\partial}{\partial \chi^b} \left[\frac{\partial \bar{\chi}^{m'}}{\partial \bar{\chi}^a} A^{\overline{a}} \right] \\ &= \frac{\partial \chi^b}{\partial \chi^{n'}} \frac{\partial \bar{\chi}^{m'}}{\partial \bar{\chi}^a} \frac{\partial A^{\overline{a}}}{\partial \chi^b}. \end{aligned}\right\} \tag{5.12}$$

In getting the last step we have made use of the fact, stated in the equation (4.1), that $\bar{\chi}^{m'}$ does not depend on χ^a so that

$$\frac{\partial}{\partial \chi^b}\left[\frac{\partial \bar{\chi}^{m'}}{\partial \bar{\chi}^a}\right] = 0. \tag{5.13}$$

From (5.12) we see that $\dfrac{\partial A^{\overline{m}}}{\partial \chi^n}$ is a tensor and thus there is no need of introducing any affinity here. Alternatively, we can say that the covariant derivative of $A^{\overline{m}}$ with respect to χ^n is the same as the ordinary derivative

$$A^{\overline{m}}_{;\,n} = \frac{\partial A^{\overline{m}}}{\partial \chi^n}. \tag{5.14}$$

We summarize here the rules for covariant differentiation:

$$\left.\begin{aligned}
\varphi_{;\,n} &= \frac{\partial \varphi}{\partial \chi^n} & \varphi_{;\,\overline{n}} &= \frac{\partial \varphi}{\partial \bar{\chi}^n} \\
A^m_{;\,n} &= \frac{\partial A^m}{\partial \chi^n} + A^a\, \Gamma^m_{an} & A^m_{;\,\overline{n}} &= \frac{\partial A^m}{\partial \bar{\chi}^n} \\
A_{m;\,n} &= \frac{\partial A_m}{\partial \chi^n} - A_a\, \Gamma^a_{mn} & A_{m;\,\overline{n}} &= \frac{\partial A_m}{\partial \bar{\chi}^n} \\
A^{\overline{m}}_{;\,n} &= \frac{\partial A^{\overline{m}}}{\partial \chi^n} & A^{\overline{m}}_{;\,\overline{n}} &= \frac{\partial A^{\overline{m}}}{\partial \bar{\chi}^n} + A^{\overline{a}}\, \Gamma^{\overline{m}}_{\overline{a}\overline{n}} \\
A_{\overline{m};\,n} &= \frac{\partial A_{\overline{m}}}{\partial \chi^n} & A_{\overline{m};\,\overline{n}} &= \frac{\partial A_{\overline{m}}}{\partial \bar{\chi}^n} - A_{\overline{a}}\, \Gamma^{\overline{a}}_{\overline{m}\overline{n}} \\
&& \Gamma^{\overline{m}}_{\overline{n}\overline{r}} &= \overline{\Gamma^m_{nr}}
\end{aligned}\right\} \tag{5.15}$$

The differentiation rules for tensors can be obtained from (5.15) and (5.6). A tensor like A_{mn} transforms like the product of two vectors B_m and C_n. Therefore its differentiation law ought to be the same as for the product $B_m\, C_n$. This gives

$$\left.\begin{aligned}
A_{mn;\,r} &= \frac{\partial A_{mn}}{\partial \chi^r} - A_{an}\,\Gamma^a_{mr} - A_{ma}\,\Gamma^a_{nr}.\\
&\text{Similarly,}\\
A^{mn}_{;r} &= \frac{\partial A^{mn}}{\partial \chi^r} + A^{an}\,\Gamma^m_{ar} + A^{ma}\,\Gamma^n_{ar}\\
A_{\overline{m}n;\,r} &= \frac{\partial A_{\overline{m}n}}{\partial \chi^r} - A_{\overline{m}a}\,\Gamma^a_{nr}\\
A^{\overline{m}}{}_{n;\,r} &= \frac{\partial A^{\overline{m}}_n}{\partial \chi^r} - A^{\overline{m}}{}_a\,\Gamma^a_{nr}\\
A^{\overline{m}n}_{;\,r} &= \frac{\partial A^{\overline{m}n}}{\partial \chi^r} + A^{\overline{m}a}\,\Gamma^n_{ar}\\
A_{\overline{m}}{}^n{}_{;\,r} &= \frac{\partial A^n_{\overline{m}}}{\partial \chi^r} + A_{\overline{m}}{}^a\,\Gamma^n_{ar}.
\end{aligned}\right\}\qquad (5.16)$$

The rules for differentiating with regard to $\overline{\chi}^r$ are quite similar. One can easily write down the differentiation rules for tensors of higher ranks.

6. Lowering and Raising of Suffixes; Relation between the Affinity and the Metric.

We use the metric $\eta_{\overline{m}n}$ to lower the indices of tensors in the following way:

$$A_n = A^{\overline{m}}\,\eta_{\overline{m}n} \qquad (6.1\text{ a})$$

and

$$A_{\overline{m}} = \eta_{\overline{m}n}\,A^n. \qquad (6.1\text{ b})$$

From (6.1 a) it follows that

$$A_{n;\,r}\; A^{\overline{m}}_{;\,r}\,\eta_{\overline{m}n} + A^{\overline{m}}\,\eta_{\overline{m}n;\,r}. \qquad (6.2)$$

We now assume that

$$\eta_{\overline{m}n;\,\mathrm{r}} = 0\,. \qquad (6.3)$$

As one can see from (6.2), this enables us to perform the operation of lowering the suffixes inside the differentiation sign, viz.,

$$A_{n;\,r} = A^{\overline{m}}_{;\,r}\,\eta_{\overline{m}n}.$$

From (5.16) it follows that the equation (6.3) means

$$\frac{\partial \eta_{\overline{m}n}}{\partial \chi^r} - \eta_{\overline{m}a}\,\Gamma^a_{nr} = 0. \tag{6.4}$$

Let us now introduce the inverse of $\eta_{\overline{m}n}$. We denote it by $\eta^{m\overline{n}}$:

$$\eta_{\overline{m}a}\,\eta^{a\overline{n}} = I^{\overline{n}}_{\overline{m}} \tag{6.5 a}$$

$$\eta^{m\overline{a}}\,\eta_{\overline{a}n} = I^m_n, \tag{6.5 b}$$

where I is the unit matrix so that I^m_n equals 1 if $m = n$ and is zero otherwise. One can easily establish the tensor character of $\eta^{m\overline{n}}$ and of I^m_n. The matrix $\eta^{m\overline{n}}$ can be used to raise the suffixes of covariant vectors and tensors in a way analogous to (6.1 a, b).

If we multiply (6.4) by $\eta^{s\overline{m}}$, we get

$$\eta^{s\overline{m}}\,\frac{\partial \eta_{\overline{m}n}}{\partial \chi^r} - I^s_a\,\Gamma^a_{nr} = 0$$

or

$$\Gamma^s_{nr} = \eta^{s\overline{m}}\,\frac{\partial \eta_{\overline{m}n}}{\partial \chi^r}. \tag{6.6}$$

We have thus expressed the affinity in terms of the fundamental metric $\eta_{\overline{m}n}$. Note that the right-hand side of (6.6) is not, in general, symmetric in n and r. If we want Γ^s_{nr} to be symmetric, we must impose some restrictions on the metric; namely, the metric has to satisfy

$$\frac{\partial \eta_{\overline{m}n}}{\partial \chi^r} = \frac{\partial \eta_{\overline{m}r}}{\partial \chi^n}. \tag{6.7 a}$$

By taking the complex conjugate of (6.7 a) and using the fact that $\eta_{\overline{m}n}$ is a Hermitian matrix [see (4.8)], we also get

$$\frac{\partial \eta_{\overline{n}m}}{\partial \overline{\chi}^r} = \frac{\partial \eta_{\overline{r}m}}{\partial \overline{\chi}^n}. \tag{6.7 b}$$

The equations (6.7 a, b) show that we can write

$$\eta_{\overline{m}n} = \frac{\partial^2 \varphi}{\partial \overline{\chi}^m \, \partial \chi^n} \tag{6.8}$$

where φ is a real local scalar.

Taking the complex conjugate of (6.6) and using (5.11) and (4.8), we find

$$\Gamma^{\overline{s}}_{\overline{n}\,\overline{r}} = \frac{\partial \eta_{\overline{n}m}}{\partial \overline{\chi}^r} \, \eta_{m\overline{s}}. \tag{6.9}$$

7. Curvature Tensor.

The expression (6.6) looks very different from the usual expression for affinity in relativity theory. However, it will be shown that, by using a suitable notation, we can put it in a form similar to that in relativity theory.

Let us define

$$\overline{\chi^m} = \chi^{N+m}. \tag{7.1}$$

In general, let us write $N+m$ instead of $\overline{m}$ wherever the latter occurs. Thus, in our new notation,

and

$$\left.\begin{aligned} A^{\overline{m}} &= A^{N+m} \\ \eta_{\overline{m}n} &= \eta_{(N+m)\,n}. \end{aligned}\right\} \tag{7.2}$$

We also define

$$\eta_{\mu\nu} = \left\{\begin{array}{ll} \eta_{\overline{(\mu-N)}\,\nu} & \text{when } \mu > N,\ \nu \leq N \\ 0 & \text{in all other cases} \end{array}\right\} \tag{7.3}$$

where, in this section, the Greek indices take the values $1,2\ldots 2\,N$.

The invariant line element (3.4) becomes

$$\sum_{\mu,\nu} d\chi^\mu \, \eta_{\mu\nu} \, d\chi^\nu = dt^2 \tag{7.4}$$

which is similar to the expression (3.5) of the relativity theory. One can easily verify that the expressions (6.6) and (6.9) for the affinity can now be written together as

$$\left.\begin{aligned}\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}\,\eta^{\mu\alpha}&\left[\frac{\partial\eta_{\alpha\nu}}{\partial\chi^{\sigma}}+\frac{\partial\eta_{\alpha\sigma}}{\partial\chi^{\nu}}-\frac{\partial\eta_{\nu\sigma}}{\partial\chi^{\alpha}}\right]\\ &+\frac{1}{2}\left[\frac{\partial\eta_{\nu\alpha}}{\partial\chi^{\sigma}}+\frac{\partial\eta_{\sigma\alpha}}{\partial\chi^{\nu}}-\frac{\partial\eta_{\sigma\nu}}{\partial\chi^{\alpha}}\right]\eta^{\alpha\mu}.\end{aligned}\right\} \tag{7.5}$$

(7.5) is quite similar to the usual expression in real space. The inverse matrix $\eta^{\mu\nu}$ is here defined as

$$\eta^{\mu\nu} = \left\{\begin{array}{ll}\eta^{\mu\,\overline{(N-\nu)}} & \text{when } \mu \leq N,\ \nu > N\\ 0 & \text{in all other cases.}\end{array}\right\} \tag{7.6}$$

As in the tensor analysis of real space, the expression (7.5) gives rise to the curvature tensor

$$B^{\mu}{}_{\nu\varrho\sigma} = -\frac{\partial\,\Gamma^{\mu}_{\nu\varrho}}{\partial\,\chi^{\sigma}}+\frac{\partial\,\Gamma^{\mu}_{\nu\sigma}}{\partial\,\chi^{\varrho}}+\Gamma^{\mu}_{\alpha\varrho}\,\Gamma^{\alpha}_{\nu\sigma}-\Gamma^{\mu}_{\alpha\sigma}\,\Gamma^{\alpha}_{\nu\varrho}. \tag{7.7}$$

If we write (7.7) in our previous notation, using barred and unbarred suffixes, we find

$$\left.\begin{aligned}B^{m}{}_{nrs} = 0,\quad B^{m}{}_{\bar{n}rs} = 0,\quad B^{m}{}_{n\bar{r}\bar{s}} = 0,\quad B^{m}{}_{\bar{n}\bar{r}s} = 0,\\ B^{m}{}_{\bar{n}r\bar{s}} = 0,\quad B^{m}{}_{\bar{n}\bar{r}s} = 0;\end{aligned}\right\} \tag{7.8}$$

$$B^{m}{}_{nr\bar{s}} = -B^{m}{}_{n\bar{s}r} = -\frac{\partial\,\Gamma^{m}_{nr}}{\partial\,\bar{\chi}^{s}}. \tag{7.9}$$

The complex conjugates of (7.8) and (7.9) also hold.

Thus $B^{m}{}_{nr\bar{s}}$ is essentially the curvature tensor in this theory[1].

8. Condition for Flat Space.

One can easily show that the affinity can be made zero at any *one* given point, say at the origin, by a suitable choice of the coordinate system. In fact, not only the affinity Γ^{m}_{nr}, but also its

[1] I am thankful to Professor C. Møller for first pointing this out to me. The expression (7.9) can also be found directly from the previous formalism (without introducing the notations (7.1)–(7.3)) by extending the idea of parallel displacements to the complex space. (Private communication).

See appendix for the contracted forms of the curvature tensor.

gradient $\frac{\partial \Gamma^m_{nr}}{\partial \chi^s}$ can be made to vanish at any *one* point. This can be explicitly verified by carrying out the transformation

$$\chi^m = \chi^{m'} + \frac{1}{2} \alpha^m_{nr} \chi^{n'} \chi^{r'} + \frac{1}{6} \beta^m_{nrs} \chi^{n'} \chi^{r'} \chi^{s'}, \tag{8.1}$$

where

$$\alpha^m_{nr} = -(\Gamma^m_{nr})_0 \tag{8.2 a}$$

and

$$\left.\begin{aligned} \beta^m_{nrs} &= \left[\Gamma^m_{na} \Gamma^a_{sr} + \Gamma^m_{ar} \Gamma^a_{sn} - \frac{\partial \Gamma^m_{nr}}{\partial \chi^s}\right]_0 \\ &= \left[\Gamma^m_{as} \Gamma^a_{nr} + \Gamma^m_{an} \Gamma^a_{sr} + \Gamma^m_{ar} \Gamma^a_{sn} - \eta^{m\bar{a}} \frac{\partial^2 \eta_{\bar{a}n}}{\partial \chi^r \partial \chi^s}\right]_0. \end{aligned}\right\} \tag{8.2 b}$$

Note that, due to the relation (6.7 a), the right-hand side in (8.2 b) is symmetric in n, r and s, as it should be because (8.1) shows that $B^m{}_{nrs}$ is symmetric in these suffixes.

However, the affinity and its gradient vanish at one point only; they do not vanish even at a neighbouring point unless the curvature tensor (7.9) vanishes. To see this, we have merely to expand Γ^m_{nr} in a Taylor series about the origin:

$$\left.\begin{aligned} \Gamma^m_{nr}(d\chi, d\bar{\chi}) &= (\Gamma^m_{nr})_0 + \left[\frac{\partial \Gamma^m_{nr}}{\partial \chi^s}\right] d\chi^s + \left[\frac{\partial \Gamma^m_{nr}}{\partial \bar{\chi}^s}\right]_0 d\bar{\chi}^s + \dots \\ &= -\left[B^m{}_{nr\bar{s}}\right]_0 d\bar{\chi}^s + \dots \end{aligned}\right\} \tag{8.3}$$

Hence, it is a necessary condition for the vanishing of the affinity that the curvature tensor must vanish:

$$B^m{}_{nr\bar{s}} = 0. \tag{8.4}$$

As the form (7.7) closely resembles the expression for the curvature tensor in real space, it is not difficult to see that (8.4) is also a sufficient condition for the vanishing of the affinity in some coordinate system.

From (6.6) and (6.9) we see that, if the affinity vanishes, $\eta_{\bar{m}n}$ are constants independent of χ, $\bar{\chi}$. In other words, the space is then flat and we can take, by correspondence with the usual quantum theory,

$$\eta_{\overline{m}n} = I_{\overline{m}n}, \tag{8.5}$$

where I is the unit matrix, i. e.,

$$I_{\overline{m}n} \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n. \end{cases} \tag{8.6}$$

From the above discussion it follows that, if our theory is to be essentially different from the usual quantum theory, we must have a curvature tensor which is not zero.

9. Equations of Motion.

In general relativity theory, the equations of motion of a particle in a gravitational field can be obtained by the variation of the Lagrangian

$$\mathfrak{L} = \int \left[g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \right]^{1/2} d\tau$$

with respect to $x^{\mu}(\tau)$. We assume that we can obtain the equations of motion for $\chi(t)$ in our quantum theory by a similar variational principle. As the Lagrangian we take

$$\left.\begin{aligned} \mathfrak{L} &= \int_{t_0}^{t_1} L\, dt \\ L &= \left[\eta_{\overline{m}n}(\chi, \overline{\chi}) \frac{d\overline{\chi}^m}{dt} \frac{d\chi^n}{dt} \right]^{1/2} \end{aligned}\right\} \tag{9.1}$$

and make in χ and $\overline{\chi}$ independent variations that vanish at the end points t_0 and t_1.

In this way we easily obtain

$$\frac{D\psi^m}{dt} = 0 \tag{9.2 a}$$

and

$$\frac{D\overline{\psi}^m}{dt} = 0, \tag{9.2 b}$$

where

$$\left.\begin{aligned}\frac{D\psi^m}{dt} &= \psi^m_{;r}\frac{d\chi^r}{dt} + \psi^m_{;\bar{r}}\frac{d\bar{\chi}^r}{dt}\\ &= \left[\frac{\partial\psi^m}{\partial\chi^r} + \psi^a\,\Gamma^m_{ar}\right]\frac{d\chi^r}{dt} + \frac{\partial\psi^m}{\partial\bar{\chi}^r}\frac{d\bar{\chi}^r}{dt}\\ &\quad[\text{ see } (5.15)\,]\\ &= \frac{d\psi^m}{dt} + \Gamma^m_{ab}\,\psi^a\,\psi^b,\end{aligned}\right\} \tag{9.3 a}$$

and similarly

$$\frac{D\bar{\psi}^m}{dt} = \frac{d\bar{\psi}^m}{dt} + \Gamma^{\overline{m}}_{\overline{a}\,\overline{b}}\,\bar{\psi}^a\,\bar{\psi}^b. \tag{9.3 b}$$

The equations (9.2 a, b) now replace the equations

$$\frac{d\varphi_m}{dt} = 0 \tag{9.4 a}$$

and

$$\frac{d\bar{\varphi}_m}{dt} = 0 \tag{9.4 b}$$

of the usual quantum theory in the Heisenberg representation. [To denote the states of the usual theory we have here used φ_m, $\bar{\varphi}_m$ to distinguish them from ψ, $\bar{\psi}$ of the present work.]

Besides (9.4 a, b), we have also equations for the dynamical variables, F:

$$\frac{dF_{mn}}{dt} = \frac{i}{\hbar}[H, F]_{mn}, \tag{9.5}$$

where the square bracket stands for the commutator. We replace these equations by the covariant ones

$$\frac{D\boldsymbol{F}^m{}_n}{dt} = \frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{F}]^m{}_n. \tag{9.6}$$

Here

$$\left.\begin{aligned}\frac{D\boldsymbol{F}^m{}_n}{dt} &= \boldsymbol{F}^m{}_{n;\,r}\frac{d\chi^r}{dt} + \boldsymbol{F}^m{}_{n;\,\bar{r}}\frac{d\bar{\chi}^r}{dt} \\ &= \left[\frac{\partial \boldsymbol{F}^m{}_n}{\partial \chi^r} + \boldsymbol{F}^a{}_n\,\Gamma^m_{ar} - \boldsymbol{F}^m{}_a\,\Gamma^a_{nr}\right]\frac{d\chi_r}{dt} + \frac{\partial \boldsymbol{F}^m{}_n}{\partial \bar{\chi}^r}\frac{d\bar{\chi}^r}{dt} \quad \text{[see (5.16)]} \\ &= \frac{d\boldsymbol{F}^m{}_n}{dt} + [\Gamma^m_{ar}\,\boldsymbol{F}^m{}_a - \boldsymbol{F}^m{}_a\,\Gamma^a_{nr}]\,\psi^r\,.\end{aligned}\right\} \quad (9.7)$$

It will be shown below that the Hamiltonian $\boldsymbol{H}$ of the new theory is, in general, different from the Hamiltonian H of the old theory. This is the reason why the two Hamiltonians have been written in different ways in (9.5) and (9.6).

It may be remarked that, when the space is flat so that the affinity vanishes, the equations (9.2 a, b) and (9.6) of the new theory reduce to the equations (9.4 a, b) and (9.5) of the old theory.

10. Relation between the Old and the New Hamiltonians.

If we replace $\boldsymbol{F}$ by $\boldsymbol{H}$ in (9.6), we get

$$\frac{D\boldsymbol{H}^m{}_n}{dt} = 0 \qquad (10.1)$$

or

$$\frac{d\boldsymbol{H}^m{}_n}{dt} = (\boldsymbol{H}^m{}_a\;\Gamma^a_{nr} - \Gamma^m_{ar}\,\boldsymbol{H}^a{}_n)\,\psi^r \qquad (10.2)$$

[see (9.7)]

or

$$\boldsymbol{H}^m{}_n(t) = \boldsymbol{H}^m{}_n(0) + \int_0^t \boldsymbol{H}^m{}_a\,\Gamma^a_{nr} - \Gamma^m_{ar}\,\boldsymbol{H}^a{}_n)\,\psi^r\,dt. \qquad (10.3)$$

Let us understand by 0 the instant at which the geodesic coordinates are introduced such that

$$\Gamma^m_{nr}\,(0) = 0, \quad \frac{\partial \Gamma^m_{nr}}{\partial \chi^s}\,(0) = 0 \qquad (10.4)$$

[see section 8]. At this instant the equations (9.6) and (9.2 a, b) of our theory go over into the equations (9.5) and (9.4 a, b) of the old theory. We can therefore put

$$\boldsymbol{H}^{m}{}_{n}\,(0) = H^{m}{}_{n}\,, \tag{10.5}$$

where $H^{m}{}_{n}$ is the old Hamiltonian. Then (10.3) becomes

$$\boldsymbol{H}^{m}{}_{n} = H^{m}{}_{n} + \int_0^t (H^{m}{}_{a}\, \Gamma^{a}_{nr} - \Gamma^{m}_{ar}\, \boldsymbol{H}^{a}{}_{n})\, \psi^r\, dt. \tag{10.6}$$

This shows that, as the affinity does not vanish everywhere on the track, the Hamiltonian $\boldsymbol{H}$ is, in general, different from the Hamiltonian H.

11. Expectation Values and Equivalence of the Old and the New Theories.

The expectation value of an observable $\boldsymbol{F}$ in this theory is given by

$$<\boldsymbol{F}> = \psi_m\, \boldsymbol{F}^{m}{}_{n}\, \psi^n. \tag{11.1}$$

We shall now show that this is the same as the expectation value

$$<F>_u = \bar{\varphi}_m\, F_{mn}\, \varphi_n\,, \tag{11.2}$$

where φ_m denotes the states of the usual theory [cf. (9.4 a, b), (9.5)] and the suffix 'u' denotes the 'usual' theory to distinguish (11.2) from (11.1). To show this, we first remark that, in general, the expectation values (11.1) and (11.2) depend on time. At the instant 0 at which we introduce the geodesic coordinates [cf. sections 9 and 10], we can take both of them to be equal:

$$<\boldsymbol{F}>(0) = <F>_u(0). \tag{11.3}$$

To get the expectation value at any later instant t, we make the Taylor expansion

$$\left.\begin{aligned} <\boldsymbol{F}>(t) &= <\boldsymbol{F}>(0) + t\left[\frac{d<\boldsymbol{F}>}{dt}\right]_0 \\ &\quad + \frac{1}{2!}\,t^2\left[\frac{d^2<\boldsymbol{F}>}{dt^2}\right]_0 + \cdots \\ &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!}\left[\frac{d^\nu<\boldsymbol{F}>}{dt^\nu}\right]_0. \end{aligned}\right\} \tag{11.4}$$

Now

$$\frac{d<\boldsymbol{F}>}{dt} = \frac{D<\boldsymbol{F}>}{dt} \text{ [because } <\boldsymbol{F}> \text{ is a scalar]}$$

$$= \psi_m \frac{D\boldsymbol{F}^m{}_n}{dt} \psi^n \text{ [using (9.2 a, b)]}$$

$$= \frac{i}{\hbar} \psi_m [\boldsymbol{H}, \boldsymbol{F}]^m{}_n \psi^n \text{ [using (9.6)].}$$

At the instant 0, all the variables of the new theory go over into those of the old theory, giving

$$\left[\frac{d<\boldsymbol{F}>}{dt}\right]_0 = \frac{i}{\hbar} \overline{\varphi}_m [H, F]_{mn} \varphi_n,$$

Similarly, one can easily see on using (9.2 a, b), (9.6), and (10.1), that

$$\left[\frac{d^\nu<\boldsymbol{F}>}{dt^\nu}\right]_0 = \left(\frac{i}{\hbar}\right)^\nu \overline{\varphi}_m [H, F]^{(\nu)}{}_{mn} \varphi_n, \tag{11.5}$$

where

$$[H, F]^{(\nu)} = \underbrace{[H, [H, \ldots\ldots [H, F]] \ldots]}_{\nu \text{ terms}}. \tag{11.6}$$

Thus (11.4) becomes

$$<\boldsymbol{F}>(t) = \sum_{\nu=0}^{\infty} \left(\frac{i}{\hbar}\right)^\nu \frac{t^\nu}{\nu!} \overline{\varphi}_m H, F]^{(\nu)}{}_{mn} \varphi_n. \tag{11.7}$$

This, however, is precisely the expression that one would obtain also from (11.2). Hence

$$<\boldsymbol{F}>(t) = <\boldsymbol{F}>_u(t).$$

Thus, the expectation values of all dynamical variables will be the same in the new theory as in the old[1]. Note that this result does not depend on the curvature tensor.

[1] I am thankful to Professor C. MØLLER for pointing out in a letter to me this equivalence of the old and the new theories.

12. Conclusion and Outlook.

We have tried to make a generalization of the Hilbert space by introducing the variables χ, $\bar{\chi}$, and admitting quite general transformations of these 'coordinates'. It was hoped that this might lead to a more general theory than the present quantum mechanics which allows only linear transformations of states. The result of the last section, however, shows that, irrespective of whether the space is curved or not, the physical results of the new theory will be the same as those obtained from the old theory. We therefore conclude that no essential generalization of quantum mechanics can be obtained, at least in the framework of the present formalism, by introducing a curved Hilbert space.

There are, however, a number of questions that need clarification and may provide further insight into the theory. The most important of them is whether we can assign any physical significance to the variables χ, $\bar{\chi}$. It would be interesting also to understand the significance of the relation (3.4) in which the arc-length in χ-space is identified with the physical time, t. Besides these questions of interpretation, there are also some mathematical points that need examination. In section 9, the equations (9.2 a, b) for the time-variation of the state vectors were derived by means of a variational principle, (9.1). However, the equation of motion, (9.6), for the dynamical variables was simply postulated as a generalization from the usual quantum theory. It would be of interest to investigate whether we can arrive at (9.6) also by means of a variational procedure. This equation is primarily responsible for the equivalence of physical results in the old and the new theories, and, therefore, an alteration here is likely to affect the conclusion that we have reached above. If, for example, there is a term containing the gradient of the Hamiltonian in (9.6), the latter will still be a possible generalization of (9.5), but the equivalence of the old and the new theories will no longer hold. Again, we have confined ourselves to the case of a symmetric affinity in this work. But, from section 6 it will be clear that a symmetric affinity does not appear to be the most natural thing to have in the complex space. It would therefore be worthwhile to investigate whether a non-symmetric affinity can lead to any new results. Finally, we have treated the case of a Hilbert space of finite dimensions. In quantum mechanics, however, we have to work in an infinite dimensional space. A generalization of this work to the latter case will be of interest, at least to the mathematician, and perhaps also to the physicist.

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Appendix.

Contracted Forms of the Curvature Tensor.

From the curvature tensor (7.9) we can, apparently, obtain two tensors of the second rank, viz.,

$$R_{\bar{s}r} = B^m{}_{mr\bar{s}} \tag{A. 1}$$

and

$$S^m{}_n = \eta^{r\bar{s}} B^m{}_{nr\bar{s}}. \tag{A. 2}$$

We here obtain explicit expressions for these tensors in terms of the metric η and show that they are essentially the same.

We first consider $R^{\bar{s}r}$.

From (6.5 b) we note that

$$\frac{\partial \eta^{m\bar{a}}}{\partial \bar{\chi}^s} \eta_{\bar{a}n} + \eta^{m\bar{a}} \frac{\partial \eta_{\bar{a}n}}{\partial \bar{\chi}^s} = 0$$

or

$$\frac{\partial \eta^{m\bar{r}}}{\partial \bar{\chi}^s} = - \eta^{n\bar{r}} \eta^{m\bar{a}} \frac{\partial \eta^{\bar{a}n}}{\partial \bar{\chi}^s}. \tag{A. 3}$$

Now

$$\left.\begin{aligned} R_{\bar{s}r} &= - \frac{\partial \Gamma^m_{mr}}{\partial \bar{\chi}^s} \text{ [from (7.9) and (A. 1)]} \\ &= - \frac{\partial}{\partial \bar{\chi}^s} \left[\eta_{m\bar{a}} \frac{\partial \eta_{\bar{a}m}}{\partial \chi^r} \right] \text{ [using (6.6)]} \\ &= - \frac{\partial \eta^{m\bar{a}}}{d\bar{\chi}^s} \frac{\partial \eta_{\bar{a}m}}{\partial \chi^r} - \eta^{m\bar{a}} \frac{\partial^2 \eta_{\bar{a}m}}{\partial \bar{\chi}^s \partial \chi^r} \\ &= \eta^{n\bar{a}} \eta^{m\bar{b}} \frac{\partial \eta_{\bar{b}n}}{\partial \bar{\chi}^s} \frac{\partial \eta_{\bar{a}m}}{\partial \chi^r} - \eta^{m\bar{a}} \frac{\partial^2 \eta_{\bar{s}r}}{\partial \bar{\chi}^a \partial \chi^m} \\ &\qquad \text{[using (A. 3) in the first (6.7 a, b)} \\ &\qquad \text{in the second]} \\ &= \eta^{n\bar{a}} \eta^{m\bar{b}} \frac{\partial \eta_{\bar{s}n}}{\partial \bar{\chi}^b} \frac{\partial \eta_{\bar{a}r}}{\partial \chi^m} - \Delta \eta_{\bar{s}r}, \end{aligned}\right\} \tag{A. 3}$$

where

$$\Delta = \eta^{a\bar{b}} \frac{\partial^2}{\partial \bar{\chi}^b \, \partial \chi^a}. \tag{A.5}$$

We can also derive an alternative expression for $R_{\bar{s}r}$ if we use

$$\eta^{m\bar{n}} = \frac{M^{\bar{n}m}}{|\eta|}, \tag{A.6}$$

where $M^{\bar{n}m}$ is the cofactor of $\eta_{\bar{n}m}$ in the determinant $|\eta_{\bar{a}b}| = |\eta|$. We first note that

$$\left.\begin{aligned} \frac{\partial |\eta|}{\partial \chi^r} &= \frac{\partial |\eta|}{\partial \eta_{\bar{a}b}} \frac{\partial \eta_{\bar{a}b}}{\partial \chi^r} \\ &= M^{\bar{a}b} \frac{\partial \eta_{\bar{a}b}}{\partial \chi^r} \\ &= |\eta| \, \eta^{b\bar{a}} \frac{\partial \eta_{\bar{a}b}}{\partial \chi^r}. \end{aligned}\right\} \tag{A. 7}$$

Now, from (6.6) we notice that

$$\eta^{b\bar{a}} \frac{\partial \eta_{\bar{a}b}}{\partial \chi^r} = \Gamma^a_{ar}.$$

Substituting this in (9.13) we find

$$\Gamma^a_{ar} = \frac{1}{|\eta|} \frac{\partial |\eta|}{\partial \chi^r} = \frac{\partial}{\partial \chi^r} [\log |\eta|]. \tag{A.8}$$

Hence

$$\left.\begin{aligned} R_{\bar{s}r} &= -\frac{\partial \Gamma^m_{mr}}{\partial \bar{\chi}^s} \\ &= -\frac{\partial^2}{\partial \chi^r \, \partial \bar{\chi}^s} [\log |\eta|]. \end{aligned}\right\} \tag{A. 9}$$

If we contract (A. 9) again, we get

$$R = \eta^{r\bar{s}} R_{\bar{s}r} = -\Delta \, [\log |\eta|]. \tag{A. 10}$$

Let us now look at (A. 2). We get

$$\begin{aligned}
\eta^{r\bar{s}} B^{m}_{nr\bar{s}} &= -\eta^{r\bar{s}} \frac{\partial \Gamma^{m}_{nr}}{\partial \bar{\chi}} \\
&= -\eta^{r\bar{s}} \frac{\partial}{\partial \bar{\chi}^{s}} \left[\eta^{m\bar{a}} \frac{\partial \eta_{\bar{a}n}}{\partial \chi^{r}} \right] \quad \text{[using (6.6)]} \\
&= -\eta^{r\bar{s}} \left[\frac{\partial \eta^{m\bar{a}}}{\partial \bar{\chi}^{s}} \frac{\partial \eta_{\bar{a}n}}{\partial \chi^{r}} + \eta^{m\bar{a}} \frac{\partial^{2} \eta_{an}}{\partial \bar{\chi}^{s} \partial \chi^{r}} \right] \\
&= -\eta^{r\bar{s}} \left[-\eta^{b\bar{a}} \eta^{m\bar{c}} \frac{\partial \eta_{\bar{c}b}}{\partial \bar{\chi}^{s}} \frac{\partial \eta_{\bar{a}n}}{\partial \chi^{r}} + \eta^{m\bar{a}} \frac{\partial^{2} \eta_{\bar{a}n}}{\partial \bar{\chi}^{s} \partial \chi^{r}} \right].
\end{aligned}$$

[using (A.3)]

Therefore

$$\begin{aligned}
\eta_{\bar{k}m} \eta^{r\bar{s}} B^{m}_{nr\bar{s}} &= -\eta^{r\bar{s}} \left[-\eta^{b\bar{a}} \frac{\partial \eta_{\bar{k}m}}{\partial \bar{\chi}^{s}} \frac{\partial \eta_{\bar{a}n}}{\partial \chi^{r}} + \frac{\partial^{2} \eta_{\bar{k}n}}{\partial \bar{\chi}^{s} \partial \chi^{r}} \right] \\
&= \eta^{b\bar{a}} \eta^{r\bar{s}} \frac{\partial \eta_{\bar{k}b}}{\partial \bar{\chi}^{s}} \frac{\partial \eta_{\bar{a}n}}{\partial \chi^{r}} - \Delta \eta_{\bar{k}n}
\end{aligned}$$

which is precisely the expression (A.4) when we replace k by s and n by r and use suitable dummy indices. Thus

$$R_{\bar{s}r} = \eta_{\bar{s}m} S^{m}{}_{r}$$

and hence the curvature tensor $B^{m}_{nr\bar{s}}$ gives rise to only one tensor of the second rank.

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